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DYNAMIC RIGIDITY OF A BEAM IN A MOVING CONTACT

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The equivalent dynamic rigidity of the Timoshenko beam resting on an elastoviscous base and interacting with a point object moving uniformly on it is studied. A generic relation for the equivalent rigidity of the beam is obtained and analyzed. The equivalent rigidity is studied as a function of the object velocity. A comparative analysis of equivalent rigidities of the Timoshenko and Bernoulli–Euler beams is performed.

Introduction. The notion of dynamic rigidity is used in the analysis of the dynamic response of mechanical systems consisting of concentrated elements (masses, springs, and viscous dampers). By definition, the dynamic viscosity of a system is the ratio of the amplitude of the harmonic force applied to the system and the complex amplitude of oscillations at the excited point. Obviously, in the analysis of steady responses of a complex linear system, the behavior of all parts of the latter may be described by the dynamic rigidity, which is a function of the physical parameters of the system and the oscillation frequency.

In this work, we study the dynamic rigidity of a distributed elastic system in a moving contact. This problem has been paid little notice, though it is undoubtedly of interest. From the practical point of view, this interest is due to the rapid development of high-velocity trains. In analyzing oscillations of these trains, it is important to know the dynamic rigidity of rails at the points of contact with moving wheels of the train [1, 2]. The study of the equivalent dynamic rigidity of a distributed system in a moving contact is important for the following reasons: (1) to ensure a given law of motion of an object contacting the distributed system, it is necessary (except for some cases) to apply an external force that will "pump" additional energy into the distributed system; (2) the oscillations of the moving contact point may excite propagating waves in the distributed system, including Doppler-anomalous ones [3]. Emission of these waves is related to the main feature of the equivalent rigidity in the moving contact. As is shown in this work, it can have an imaginary part corresponding to "negative viscosity," which may cause instability of oscillations of an object moving over a distributed system [4–6].

Formulation of the Problem and Generic Expression for Dynamic Rigidity. We consider a constant-velocity nonseparated motion of a point mass along the Timoshenko beam that rests on a viscoelastic base whose rigidity is uniformly distributed along the beam (Fig. 1a). Figure 1b shows the equivalent model of the Timoshenko beam.

The equations of small oscillations of the system considered have the form [7, 8]

$$\rho F U_{tt} - \chi G F U_{xx} + \chi G F \varphi_x + kU + \nu U_t = -\delta(x - Vt) m \frac{d^2 U^0}{dt^2}, \qquad (1)$$

$$\rho I \varphi_{tt} - IE\varphi_{xx} + \chi G F(\varphi - U_x) = 0, \qquad U^0(t) = U(Vt, t),$$

where $U(x,t)$ and $U^0(t)$ are the vertical deviations of the beam and the mass m, respectively, $\varphi(x,t)$ is the angle of turning of the beam cross section, E and G are the extension and shear moduli, ρ and I are the

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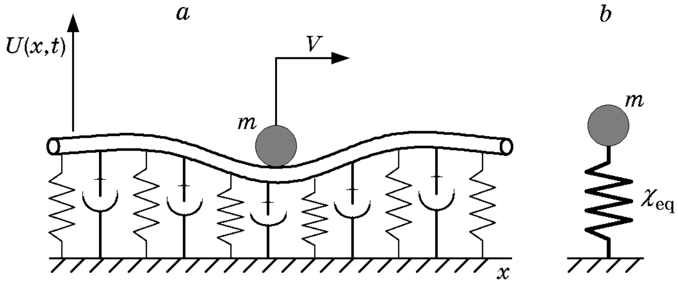


Fig. 1

density and moment of inertia of the cross section, F is the cross-sectional area of the beam, $\chi \approx 0.82$ is the Timoshenko coefficient, and k and ν are the running rigidity and viscosity of the base.

We introduce the following dimensionless variables and parameters: $c_p^2 = E/\rho$, $c^2 = G/\rho$, $c_s^2 = \chi c^2$, $\eta = c_s^2 F/(\omega^2 I)$, $\gamma = c_p^2/c_s^2$, $\alpha = V/c_s$, $\nu_1 = \nu/(\rho F \omega)$, $M = m\omega/(\rho F c_s)$, $U = U\omega/c_s$, $\tau = \omega t$, and $y = x\omega/c_s$.

From (1) we obtain

$$U_{\tau\tau} - U_{yy} + \varphi_y + U + \nu_1 U_\tau = -M\delta(y - \alpha\tau)\,\frac{d^2U^0}{d\tau^2},$$
$$\varphi_{\tau\tau} - \gamma\varphi_{yy} + \eta(\varphi - U_y) = 0, \qquad U^0(\tau) = U(V\tau,\tau). \tag{2}$$

For subsequent analysis, it is convenient to introduce a moving coordinate system ($\xi = y - \alpha\tau$, $\tau = \tau$), where Eqs. (2) take the following form:

$$U_{\tau\tau} - 2\alpha U_{\tau\xi} + (\alpha^2 - 1)U_{\xi\xi} + \varphi_\xi + U + \nu_1 U_\tau - \nu_1\alpha U_\xi = -M\delta(\xi)U_{\tau\tau}(0,\tau),$$
$$\varphi_{\tau\tau} - 2\alpha\varphi_{\tau\xi} + (\alpha^2 - \gamma)\varphi_{\xi\xi} + \eta(\varphi - U_\xi) = 0. \tag{3}$$

As follows from (3), passing to the moving coordinate system, we eliminated the mass displacement U^0. The delta function depends only on the spatial variable ξ, which allows us to analyze the stability of the system in the Laplace image space without the inverse transform.

Using the Fourier transform for the variables ξ and τ

$$\{\tilde U_\omega(\xi,\omega), \tilde\varphi_\omega(\xi,\omega)\} = \int\limits_{-\infty}^{\infty} \{U(\xi,\tau), \varphi(\xi,\tau)\} \exp(-i\omega\tau)\,d\tau,$$

$$\{\tilde{\tilde U}_{k,\omega}(k,\omega), \tilde{\tilde\varphi}_{k,\omega}(k,\omega)\} = \int\limits_{-\infty}^{\infty} \{\tilde U_\omega(\xi,\omega), \tilde\varphi_\omega(\xi,\omega)\} \exp(-ik\xi)\,d\xi,$$

we pass from the system of partial differential equations (3) to the system of algebraic equations in $\tilde{\tilde U}_{k,\omega}$ and $\tilde{\tilde\varphi}_{k,\omega}$; eliminating $\tilde{\tilde\varphi}_{k,\omega}$ from this system, we obtain

$$\tilde{\tilde U} D_{\rm T}(k,\omega) = M\omega^2 \tilde U(0,\omega), \quad D_{\rm T}(k,\omega) = [A(k,\omega)B(k,\omega) - \eta k^2]/A(k,\omega),$$
$$A(k,\omega) = -\omega^2 + 2\alpha k\omega - k^2(\alpha^2 - \gamma) + \eta, \tag{4}$$
$$B(k,\omega) = -\omega^2 + 2\alpha k\omega - k^2(\alpha^2 - 1) + \nu_1 i\omega - \nu_1\alpha i k + 1.$$

To find the closed equation for the Fourier image of the mass displacement $\tilde U(0,\omega)$, we apply the inverse Fourier transform with respect to the wavenumber k to Eq. (4). Assuming that $\xi = 0$ in the resultant equation, we obtain the equation that describes the vertical oscillations of the mass moving along the beam without separation:

$$\tilde U(0,\omega)(-M\omega^2 + \chi_{\rm eq}(\omega,V)) = 0, \quad \chi_{\rm eq}(\omega,V) = \Big(\frac{1}{2\pi}\int\limits_{-\infty}^{\infty} \frac{dk}{D_{\rm T}(k,\omega)}\Big)^{-1}. \tag{5}$$

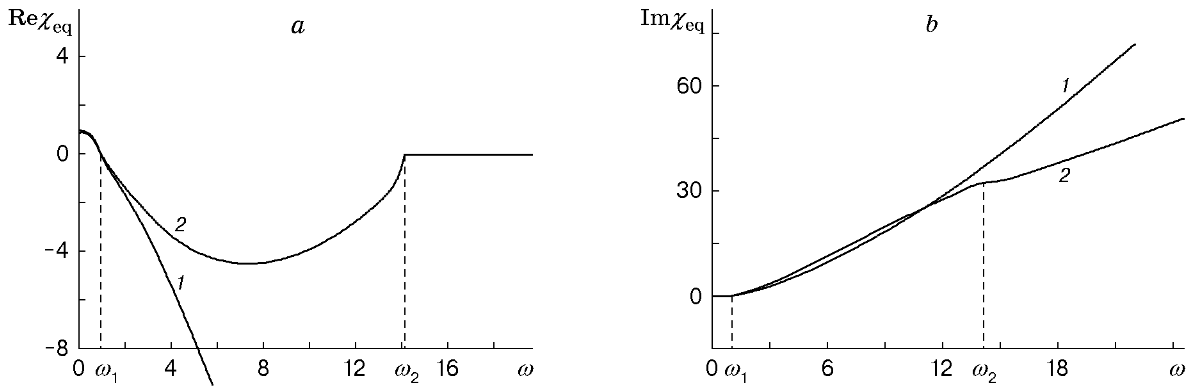


Fig. 2

The equation $-M\omega^2+\chi_{\mathrm{eq}}(\omega, V) = 0$ is the characteristic equation of oscillations of the mass moving uniformly along the beam. Obviously, the second term in this equation is the vertical dynamic rigidity of the Timoshenko beam at the contact point (see Fig. 1b).

In further analysis of the dynamic rigidity of the Timoshenko beam, we compare it with the rigidity of the Bernoulli–Euler beam, which will allow us to estimate the applicability of this simpler model for studying the dynamic behavior of objects moving along the beam.

The equations of oscillations of the Bernoulli–Euler beam interacting with a mass moving along it without separation have the following form [8]:

$$\rho F U_{tt} + EIU_{xxxx} + kU + \nu U_t = -\delta(x - Vt)m\,\frac{d^2U^0}{dt^2}, \quad U^0(t) = U(Vt, t). \tag{6}$$

The expression for the dynamic rigidity of the beam may be written as follows:

$$\chi_{\mathrm{eq}}(\omega, V) = \left(\frac{1}{2\pi}\int\limits_{-\infty}^{\infty}\frac{dk}{D_{\mathrm{BE}}(k,\omega)}\right)^{-1},$$

$$D_{\mathrm{BE}}(k,\omega) = \gamma k^4/\eta - \alpha^2k^2 - (\nu_1 i\alpha - 2\alpha\omega)k + \nu_1 i\omega - \omega^2 + 1. \tag{7}$$

Comparison of Dynamic Rigidities of the Timoshenko and Bernoulli–Euler Beams. Studying relations (5) and (7), we calculate the integrals in these relations by the method of contour integration. The integrands in these relations are fractions (with a fourth-power polynomial in the denominator) whose roots are simple poles. According to the residue theory [9], the calculation result for the integrals may be written in the form

$$\chi^{\mathrm{T}}_{\mathrm{eq}}(\omega, V) = \left(i\sum_{j=1}^{4}\frac{A(\omega, k_j)}{\partial(A(k,\omega)B(k,\omega) - \eta k^2)/\partial k\Big|_{k=k_j}}\right)^{-1},$$

$$\chi^{\mathrm{BE}}_{\mathrm{eq}}(\omega, V) = \left(i\sum_{n=1}^{4}\frac{1}{\partial D_{\mathrm{BE}}(k,\omega)/\partial k\Big|_{k=k_n}}\right)^{-1},$$

where k_j and k_n are the roots of the equations $A(k,\omega)B(k,\omega) - \eta k^2 = 0$ and $D_{\mathrm{BE}}(k,\omega) = 0$, respectively, lying in the upper half-plane of the complex variable k.

Thus, to construct the dependence of the dynamic rigidity of the beams on frequency, one has to know only the roots of these equations, which can be found numerically using the standard procedure for finding complex roots of the polynomial.

Figure 2a and b shows the dependences of the real and imaginary parts of $\chi^{\mathrm{T}}_{\mathrm{eq}}(\omega)$ and $\chi^{\mathrm{BE}}_{\mathrm{eq}}(\omega)$ for the case of stationary loading ($\alpha = 0$). The dimensionless parameters $\gamma = 2.89$, $\eta = 200$, and $\nu_1 = 0.05$ were

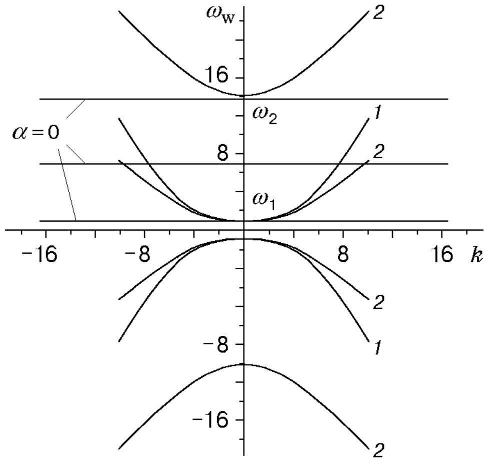


Fig. 3

used in the calculations. It is seen in Fig. 2 that there are two critical values of frequency ω_1 and ω_2 above which the behavior of $\chi_{\rm eq}^{\rm T}(\omega)$ (curves 2) changes [the bifurcation frequency for $\chi_{\rm eq}^{\rm BE}(\omega)$ (curves 1) is only ω_1, which is equal to unity as the base viscosity tends to zero). In the frequency interval $\omega < \omega_1$, the imaginary part of the rigidities is close to zero, and the real part is positive, i.e., the response of the beam is close to a purely elastic one. In passing through the frequency ω_1, the imaginary part of the equivalent rigidities rapidly increases, and the real part changes its sign, i.e., these functions become significantly complex-valued; their imaginary parts reflect the dissipative character of the beam response, and their real parts, being negative, reflect the inertial character of the beam response. Reaching the frequency ω_2, the real part of $\chi_{\rm eq}^{\rm T}(\omega)$ vanishes, and the imaginary part continues to grow. Hence, the response of the Timoshenko beam in this frequency range is purely viscous in the sense that the contact force is equivalent to the response of a damper with a frequency-dependent viscosity.

The physical meaning of the critical frequencies ω_1 and ω_2 can be easily understood if we consider the kinematics of waves excited by loading of the form $P\exp(i\omega t)$ moving along the beam with a velocity V [10]. It is known that the wavenumbers k and frequencies $\omega_{\rm w}$ of waves excited by this loading satisfy the relation $\omega_{\rm w} - kV = \omega$, which is called the kinematic invariant [10] and means that the phase of the emitted waves is equal to the phase of loading oscillations. In addition, k and $\omega_{\rm w}$ are related via a dispersion expression that may be obtained by substituting the vertical deviation of the beam and the angle of turning of its cross section into Eqs. (1) and (6) in the form $U(x,t) = A\exp[i(\omega_{\rm w}t - kx)]$ and $\varphi(x,t) = B\exp[i(\omega_{\rm w}t - kx)]$. For the Bernoulli–Euler and Timoshenko beams, we obtain

$$k^4\gamma - \eta\omega_{\rm w}^2 + \eta = 0, \quad \omega_{\rm w}^4 + (-\gamma k^2 - \eta - 1 - k^2)\omega_{\rm w}^2 + \gamma k^4 + \gamma k^2 + \eta = 0.$$

The dispersion equations determine some curves in the plane $(k, \omega_{\rm w})$, and the kinematic invariant corresponds to the straight line intersecting the axis $\omega_{\rm w}$ at the point $\omega_{\rm w} = \omega$. The angle between this line and the wavenumber axis is determined by the object velocity $V = \tan\alpha$. Fixing the velocity V (slope of the straight line) and varying the frequency of loading oscillations ω, we can analyze the dependence of the kinematic characteristics of the emitted waves on the frequency of loading oscillations. This analysis allows us to understand the behavior of the dynamic rigidities $\chi_{\rm eq}^{\rm T}(\omega)$ and $\chi_{\rm eq}^{\rm BE}(\omega)$, which are ratios of the loading amplitude to the complex amplitude of beam displacement at the contact point.

Figure 3 shows the dispersion curves for the Bernoulli–Euler (curves 1) and Timoshenko beams (curves 2) for different frequencies ω and the straight lines corresponding to the kinematic invariant for $\alpha = 0$. For critical frequencies ω_1 and ω_2, the straight lines corresponding to the kinematic invariant touch one of the dispersion curves. For $0 < \omega < \omega_1$, these dispersion curves and straight lines have no common points. Hence, waves are not excited in the beams, and energy dissipation at the contact point is caused only by losses due to the viscosity of the elastic base. We assume that these losses are rather low; therefore, the imaginary

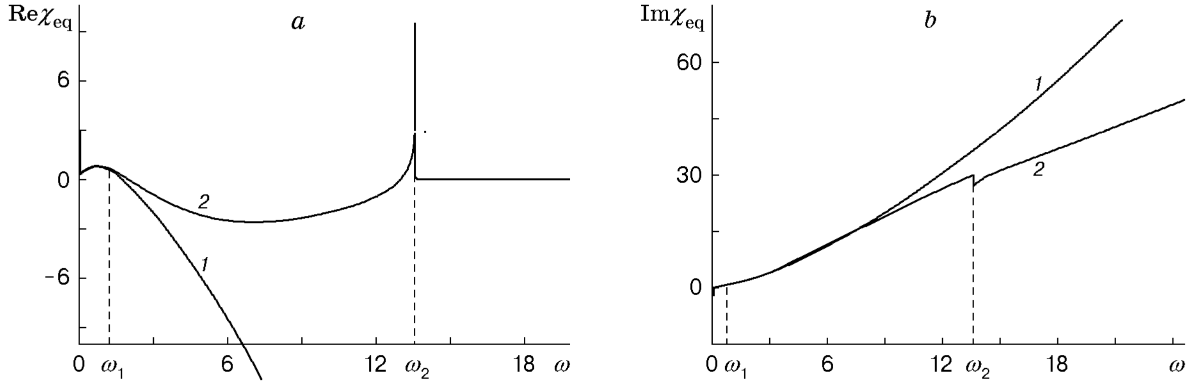


Fig. 4

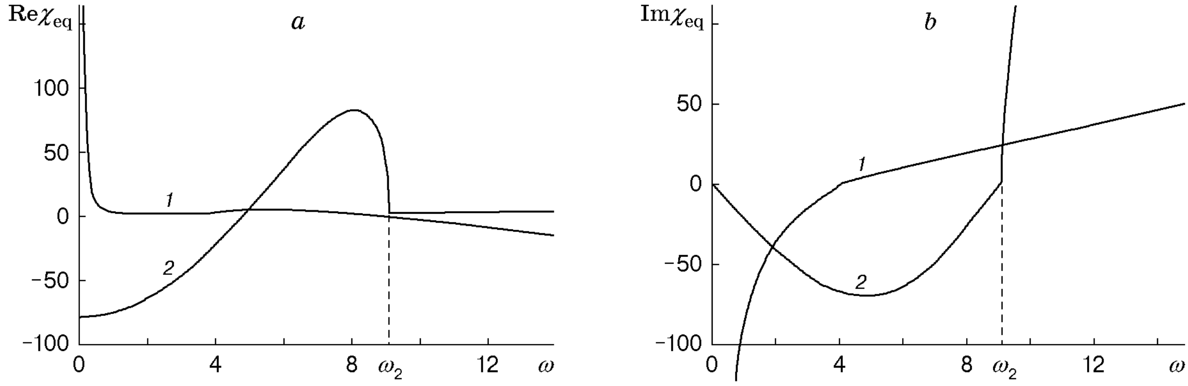


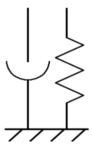
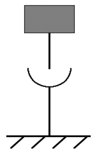
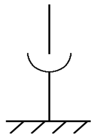
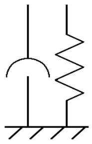
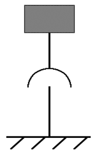
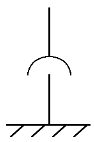
Fig. 5

parts of the equivalent rigidities are very close to zero. For $\omega \approx \omega_1$, the system experiences a resonance (the beam oscillates as a solid body), i.e., the finite amplitudes of perturbations correspond to an infinitely large displacement (the viscosity of the base is ignored). In this case, obviously, the equivalent rigidity, which is the ratio of the force and displacement, vanishes. For $\omega_1 < \omega < \omega_2$, two waves are excited in the beams, which conditions energy removal from the contact point and introduces a significant imaginary part to the equivalent rigidity. With further increase in frequency, the situation for the Bernoulli–Euler beam does not change qualitatively. For the Timoshenko beam, there exists another critical frequency ω_2, above which the field of beam displacements becomes purely wavy, which causes the viscous character of the response of the beam.

Figure 4a and b shows the dependences of the real and imaginary parts of $\chi_{\rm eq}^{\rm T}(\omega)$ and $\chi_{\rm eq}^{\rm BE}(\omega)$ for $\alpha = 0.5$ (hereinafter, the rest of the parameters are the same as for $\alpha = 0$). (The notation of curves in Figs. 4 and 5 is the same as in Fig. 2.) The difference in curves for $\alpha = 0$ and $\alpha = 0.5$ is that, in approaching the frequency ω_2, the real part of the curves $\chi_{\rm eq}^{\rm T}(\omega)$ corresponding to $\alpha = 0.5$ changes sign and becomes positive, i.e., there appears a narrow range of oscillation frequencies where the response of the Timoshenko beam has a viscoelastic character. Note that the values of ω_1 and ω_2 are different for $\alpha = 0.5$ and $\alpha = 0$. No waves are excited for $\omega < \omega_1$, two waves are emitted for $\omega_1 < \omega < \omega_2$, and four waves are excited for $\omega > \omega_2$ (only in the Timoshenko beam).

Figure 5a and b shows the dependences of the real and imaginary parts of the dynamic rigidities of the beams on frequency for $\alpha = 1.4$. The distinguishing feature of this case is the existence of the frequency range $\Delta\omega_{us}$, in which the imaginary parts of equivalent rigidities are negative. In other words, the response of the beam in a moving contact is equivalent to the response of an elastoviscous element whose viscosity depends on frequency and becomes negative for $\omega \in \Delta\omega_{us}$. From the physical viewpoint, this means that the beam response in this frequency range has the same direction as the velocity of vertical displacement of the object.

TABLE 1

$\mathrm{Re}\,\chi_{\mathrm{eq}}$	$\mathrm{Im}\,\chi_{\mathrm{eq}}$	**Model**	$\mathrm{Re}\,\chi_{\mathrm{eq}}$	$\mathrm{Im}\,\chi_{\mathrm{eq}}$	**Model**	$\mathrm{Re}\,\chi_{\mathrm{eq}}$	$\mathrm{Im}\,\chi_{\mathrm{eq}}$	**Model**
> 0	> 0		< 0	> 0		0	> 0	
> 0	< 0		< 0	< 0		0	< 0	

Harmonic oscillations of the object with frequencies $\omega \in \Delta\omega_{us}$ are unstable (the oscillation frequency starts to increase). The reason for this instability, which is well known in physics and acoustics [3, 11] and which was first described in [4] as applied to mechanical systems, is the emission of Doppler-anomalous waves (with a "negative frequency"). These waves are excited if the emitter moves with a velocity greater than the least phase velocity of waves in the medium and have the following distinctive feature: decelerating the emitter, the waves increase the internal energy of the latter. For mechanical systems, this means that the amplitude of oscillations of a moving object increases with time. The mechanism of this growth is explained in detail in [5].

We note the principal moments related to the "negative viscosity" of the beam at the contact point $[\mathrm{Im}\,\chi^{\mathrm{T}}_{\mathrm{eq}}(\omega) < 0]$ and the instability of oscillations of the moving object.

1. The frequency range in which the "negative viscosity" appears is only the necessary condition of instability. The sufficient condition of instability is the object-oscillation eigenfrequencies whose imaginary part is negative. These frequencies depend not only on the beam parameters and object velocity but also on the elastoinertial properties of the object and should be found for each particular case. The most suitable method for analysis of oscillation eigenfrequencies of an object moving over a distributed elastic system, in our opinion, is the method of D-splitting [12] applied to this class of problems in [4–6].

2. If instability occurs, then the energy necessary to "swing" the oscillations comes from a source (horizontal force applied to the object) supporting the uniform motion of the object. Note that this force differs from zero for all velocities of the object due to the viscosity of the beam base. Nevertheless, the work of this force may be directed to increasing vertical oscillations of the object only if the object exceeds the minimum phase velocity of waves in the beam.

Returning to comparison of the rigidity of the Timoshenko and Bernoulli–Euler beams, we should note that they are qualitatively different for $\alpha = 1.4$. Hence, the simpler model of the Bernoulli–Euler beam is not used in calculating oscillations of a "supercritically" moving object.

If the object velocity is greater than the velocity of longitudinal waves in the Timoshenko beam ($V > c_p \Leftrightarrow \alpha > \sqrt{\gamma} = \sqrt{2.89}$), the equivalent rigidity of the Bernoulli–Euler beam remains qualitatively unchanged, and the rigidity of the Timoshenko beam turns to infinity. The reason is that, for $V > c_p$, all the waves emitted by the moving object propagate behind it. The beam ahead of the object and beneath it is undisturbed. Hence, the nonzero force acting on the beam from the object corresponds to zero displacement of the beam, which is equivalent to $\chi^{\mathrm{T}}_{\mathrm{eq}}(\omega) \to \infty$.

It follows from the above analysis that the beam response to a harmonic action performed at the contact point is equivalent to the response of a concentrated element whose dynamic rigidity is a complex-valued function of the perturbation frequency and the velocity of this point. The real and imaginary parts of dynamic rigidity determine the elastoinertial and viscous properties of the beam response, respectively. Possible equivalent models of the beam response depending on the sign of $\mathrm{Re}\,\chi_{\mathrm{eq}}$ and $\mathrm{Im}\,\chi_{\mathrm{eq}}$ are shown in Table 1. The positive and negative real parts of rigidity are simulated by the spring and mass, respectively.

The presence of the imaginary part of dynamic rigidity is simulated by the damper. To emphasize the possibility of appearance of "negative viscosity," the damper is shown to be turned over for $\mathrm{Im}\,\chi_{\mathrm{eq}} < 0$. Note that the properties of each element depend on the oscillation frequency and velocity of the contact point.

The main conclusion of the present work is that the dynamic viscosity of the beam at a moving contact point may be negative. The presence of the range of oscillation frequencies in which the viscosity is negative is the necessary condition for the appearance of instability of oscillations of the object moving along the beam. Hence, knowing the dynamic rigidity of the beam, it is possible to state *a priori* whether instability (a mode undesirable in practical applications) can occur in various systems, such as a moving wheel of a train and a rail or a moving pantograph and a hanger.

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